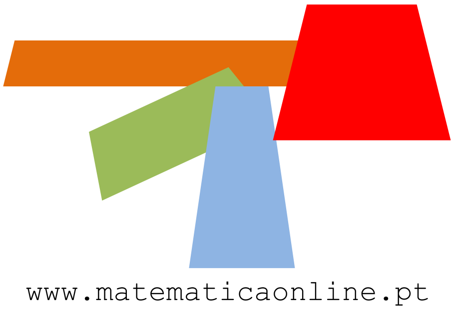
Trapézios

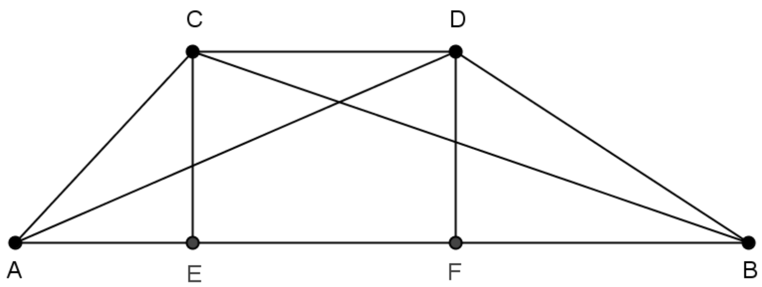
de *Só… Problemas II, João de Sacadura Cabral, 1993*

Prove que em qualquer trapézio a soma dos quadrados das diagonais é igual à soma dos quadrados dos lados oblíquos mais duas vezes o produto das bases.



Resolução

Comecemos por considerar um trapézio de bases *[AB]* e *[DC]*, tal como ilustra a figura seguinte.



Pela leitura do enunciado, temos de provar:

$$\overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + 2\overline{AB}.\overline{CD}$$

Recorrendo ao Teorema de Pitágoras, temos:

$\overline{AD}^2 = \overline{DF}^2 + \overline{AF}^2$

$\overline{BC}^2 = \overline{CE}^2 + \overline{BE}^2$

$\overline{AC}^2 = \overline{AE}^2 + \overline{CE}^2 \Leftrightarrow \overline{CE}^2 = \overline{AC}^2 - \overline{AE}^2$

$\overline{BD}^2 = \overline{BF}^2 + \overline{DF}^2 \Leftrightarrow \overline{DF}^2 = \overline{BD}^2 - \overline{BF}^2$

Podemos agora desenvolver a soma dos quadrados das diagonais.

$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{DF}^2 + \overline{AF}^2 + \overline{CE}^2 + \overline{BE}^2$$
$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{BD}^2 - \overline{BF}^2 + \overline{AF}^2 + \overline{AC}^2 - \overline{AE}^2 + \overline{BE}^2$$
$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + \overline{AF}^2 - \overline{BF}^2 + \overline{BE}^2 - \overline{AE}^2$$
$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + \left(\overline{AF} - \overline{BF}\right)\left(\overline{AF} + \overline{BF}\right) + \left(\overline{BE} - \overline{AE}\right)\left(\overline{BE} + \overline{AE}\right)$$

$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + \overline{AB}\left(\overline{AF} - \overline{BF}\right) + \overline{AB}\left(\overline{BE} - \overline{AE}\right)$$

$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + \overline{AB}\left(\overline{AF} - \overline{AE} + \overline{BE} - \overline{BF}\right)$$

$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + \overline{AB}\left(\overline{CD} + \overline{CD}\right)\overline{AD}^2 + \overline{BC}^2$$

$$\Leftrightarrow \overline{AD}^2 + \overline{BC}^2 = \overline{AC}^2 + \overline{BD}^2 + 2\overline{AB}.\overline{CD}$$